ON THE DISTRIBUTION OF ANGLES BETWEEN GEODESIC RAYS ASSOCIATED WITH HYPERBOLIC LATTICE POINTS

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ABSTRACT. For every two points z_0, z_1 in the upper-half plane $\mathbb{H}$, consider all elements γ in the principal congruence group $\Gamma(N)$, acting on $\mathbb{H}$ by fractional linear transformations, such that the hyperbolic distance between z_1 and γz_0 is at most $R > 0$. We study the distribution of angles between the geodesic rays $[z_1, \gamma z_0]$ as $R \to \infty$, proving that the limiting distribution exists independently of N and explicitly computing it. When $z_1 = z_0$ this is found to be the uniform distribution on the interval $\big[-\frac{\pi}{2}, \frac{\pi}{2}\big]$.

1. Introduction

In this paper the group $SL_2(\mathbb{R})$ acts on the upper half-plane $\mathbb{H}$ by linear fractional transformations $z \mapsto gz = \frac{az+b}{cz+d}$, $g = \big(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\big) \in SL_2(\mathbb{R})$, $z \in \mathbb{H}$. The hyperbolic ball $B(z_0, R) = \{z \in \mathbb{H} : \varrho(z_0, z) \leq R\}$ of center $z_0 = x_0 + iy_0 \in \mathbb{H}$ and radius R coincides with the Euclidean ball of center $x_0 + iy_0 \cosh R$ and radius $y_0 \sinh R \sim \frac{1}{2} y_0 e^R$. Let Γ be a discrete subgroup of $SL_2(\mathbb{R})$. The *hyperbolic circle problem* of estimating for fixed $z_0, z_1 \in \mathbb{H}$ and $R \to \infty$ the cardinality of the set $\Gamma_{z_0,R} = \{\gamma \in \Gamma : \gamma z_0 \in B(z_0, R)\}$, or slightly more generally of $\{\gamma \in \Gamma : \varrho(\gamma z_0, z_1) \leq R\}$, has been thoroughly studied with various methods (see, e.g., [4, 6, 7, 8, 9, 10, 13], and [10, 11, 12] for some higher dimensional analogs of the problem).

We consider another natural problem concerning the distribution of hyperbolic lattice points in angular sectors. For $z_0, z_1 \in \mathbb{H}$ and $g \in SL_2(\mathbb{R})$, let $\theta_{z_0,z_1}(g) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ denote the angle between the geodesic ray $[z_1, gz_0]$ and the vertical geodesic $[z_1, \infty]$. Given a compact set $\Omega \subset \mathbb{H}$ and a number $\omega \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, the proportion of points in the Γ-orbit of z_0 inside Ω such that $\theta_{z_0,z_1}(\gamma) \leq \omega$ is given by

$$\mathbb{P}_{\Gamma,\Omega,z_0,z_1}(\omega) = \frac{\#\{\gamma \in \Gamma : \gamma z_0 \in \Omega,\ \theta_{z_0,z_1}(\gamma) \leq \omega\}}{\#\{\gamma \in \Gamma : \gamma z_0 \in \Omega\}}.$$

It is natural to investigate the existence of the limiting distribution

$$\mathbb{P}_{\Gamma,z_0,z_1}(\omega) = \lim_{R\to\infty} \mathbb{P}_{\Gamma,B(z_0,R),z_0,z_1}(\omega) = \lim_{R\to\infty} \frac{\#\{\gamma \in \Gamma_{z_0,R} : \theta_{z_0,z_1}(\gamma) \leq \omega\}}{\#\Gamma_{z_0,R}}.$$

In this paper we consider the case where

$$\Gamma = \Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : a, d \equiv 1,\ b, c \equiv 0 \pmod N \right\}$$

is the *principal congruence subgroup of level* N, which is the kernel of the natural surjective morphism $SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}_N)$. This is a normal subgroup of $\Gamma(1) = SL_2(\mathbb{Z})$ of index

$$[\Gamma(1) : \Gamma(N)] = N^3 \prod_{\substack{p \mid N \\ p \text{ prime}}} (1 - p^{-2}). \tag{1.1}$$

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For every $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in SL_2(\mathbb{R})$ the hyperbolic distance $\varrho(i, gi)$ is given by

$$\cosh \varrho(i, gi) = 1 + \frac{|i - gi|^2}{2\,\mathrm{Im}(gi)} = \frac{A^2 + B^2 + C^2 + D^2}{2}. \tag{1.2}$$

Denote

$$C_N = \sum_{\substack{n \geq 1 \\ (n,N)=1}} \frac{\mu(n)}{n^2} = \prod_{p \nmid N} (1 - p^{-2}) = \frac{1}{\zeta(2)} \prod_{p \mid N} (1 - p^{-2})^{-1}. \tag{1.3}$$

For every $z_0 = x_0 + iy_0, z_1 = x_1 + iy_1 \in \mathbb{H}$, denote $x_* = \frac{x_1 - x_0}{y_0}$, $y_* = \frac{y_1}{y_0}$, and consider the continuous function Ξ_{x_*,y_*} on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ defined by

$$\begin{aligned} \Xi_{x_*,y_*}(\omega) =& \frac{1}{\pi} \arctan\left(x_* + y_* \tan \frac{\omega}{2}\right) + \frac{1}{\pi} \arctan\left(x_* - y_* \cot \frac{\omega}{2}\right) \\ & - \frac{1}{\pi} \arctan(x_* + y_*) - \frac{1}{\pi} \arctan(x_* - y_*) + \begin{cases} 1 & \text{if } \omega > 0, \\ 0 & \text{if } \omega < 0. \end{cases} \end{aligned} \tag{1.4}$$

The main result of this paper is

Theorem 1. *For every positive integer N and $z_0 = x_0 + iy_0, z_1 = x_1 + iy_1 \in \mathbb{H}$, as $R \to \infty$,*

$$\#\left\{\gamma \in \Gamma(N)_{z_0,R} : -\frac{\pi}{2} \leq \theta_{z_0,z_1}(\gamma) \leq \omega\right\} = \frac{\pi^2 C_N \Xi_{x_*,y_*}(\omega)}{N^3} e^R + O_{\varepsilon,N,z_0,z_1}\left(e^{(7/8+\varepsilon)R}\right). \tag{1.5}$$

In particular the limiting distribution $\mathbb{P}_{\Gamma(N),z_0,z_1}$ exists and is given by

$$\mathbb{P}_{\Gamma(N),z_0,z_1}(\omega) = \frac{1}{\pi} \int_{-\pi/2}^{\omega} \varrho_{z_0,z_1}(t)\, dt, \qquad \omega \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

where

$$\varrho_{z_0,z_1}(t) = \frac{2y_0 y_1 \big(y_0^2 + (x_1 - x_0)^2 + y_1^2\big)}{\big(y_0^2 + (x_1 - x_0)^2 + y_1^2\big)^2 - \big((y_0^2 + (x_1 - x_0)^2 - y_1^2) \cos t + 2(x_1 - x_0) y_1 \sin t\big)^2}.$$

Taking $z_1 = z_0$ we infer

Corollary 1. *The angles $\theta_{z_0,z_0}(\gamma)$, $\gamma \in \Gamma(N)_{z_0,R}$, are uniformly distributed as $R \to \infty$.*

The converse is also seen to be true, so that the angles $\theta_{z_0,z_1}(\gamma)$ are uniformly distributed as $R \to \infty$ if and only if $z_1 = z_0$. In the Euclidean situation these angles are uniformly distributed regardless of the choice of z_1 and z_0.

Our method of proof is number theoretical and relies on the Weil bound for Kloosterman sums [16], as previously used (for instance) in [1, 2, 3, 5, 8]. In the process we also derive, as a consequence of the proof of Theorem 1, an asymptotic formula for the number of hyperbolic lattice points in large balls.

Corollary 2. *For every positive integer N and every $z_0 \in \mathbb{H}$, as $R \to \infty$,*

$$\#\Gamma(N)_{z_0,R} = \frac{6e^R}{[\Gamma(1) : \Gamma(N)]} + O_{\varepsilon,N,z_0}\left(e^{(7/8+\varepsilon)R}\right). \tag{1.6}$$

Denoting by μ the hyperbolic area in $\mathbb{H}$, the main term in (1.5) is $\sim \frac{2\mu(B(z_0,R))}{\mu(\Gamma(N)\backslash\mathbb{H})}$ as $R \to \infty$.

For $N = 1$ formula (1.6) has been proved using Kloosterman sum estimates in [8]. Better error terms with exponent as low as $\frac{2}{3}$ can be obtained using Selberg's theory on the spectral decomposition of $L^2(\Gamma(N)\backslash\mathbb{H})$ (see [13] for exponent $\frac{3}{4}$ and [9] for exponent $\frac{2}{3}$) and lower bounds for the first eigenvalue of the Laplacian on $\Gamma(N)\backslash\mathbb{H}$ (see [14], [9], and [15] for a review of recent developments). Similar results hold when $\Gamma(N)$ is replaced by any of the congruence groups

$\Gamma_0(N)=\{\gamma\in\Gamma(1): c\equiv 0 \,(\mathrm{mod}\ N)\}$ or $\Gamma_1(N)=\{\gamma\in\Gamma(1): a,d\equiv 1, c\equiv 0\,(\mathrm{mod}\ N)\}$, or when $\Gamma(N)_{z_0,R}$ is replaced by $\{\gamma\in\Gamma(N):\varrho(\gamma z_0,z_1)\le R\}$ for fixed $z_0,z_1\in\mathbb{H}$.

There are two natural problems that arise in this context. It would be interesting to know how large is the class of discrete subgroups of $SL_2(\mathbb{R})$ for which the analogue of Theorem 1 holds. It would also be interesting to study the spacing statistics (both consecutive spacings and correlations) of these angles when $z_0=z_1$.

2. Reducing the problem to a counting problem

Given $z_0=x_0+iy_0\in\mathbb{H}$ and $\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma(1)$, consider

$$g_0=\begin{pmatrix}\sqrt{y_0} & \frac{x_0}{\sqrt{y_0}}\\ 0 & \frac{1}{\sqrt{y_0}}\end{pmatrix},\quad g=g_0^{-1}\gamma g_0=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\ \in\ SL_2(\mathbb{R}),$$

with

$$A=a-cx_0,\quad B=\frac{(a-cx_0)x_0+b-dx_0}{y_0},\quad C=cy_0,\quad D=cx_0+d. \tag{2.1}$$

Since $g_0 i=z_0$ we have

$$\cosh\varrho(z_0,\gamma z_0)=\cosh\varrho(g_0 i,g_0 g i)=\cosh\varrho(i,gi)=\frac{A^2+B^2+C^2+D^2}{2}. \tag{2.2}$$

Take $Q^2=2\cosh R\sim e^R$. As a result of (2.2) we are interested in those $\gamma\in\Gamma(N)$ for which $A^2+B^2+C^2+D^2\le Q^2$. The only matrices $\gamma\in\Gamma(1)$ with $c=0$ are $\pm I_2$ and as a result we can assume next that $C\neq 0$. We will also assume that $A\neq 0$.

The geodesic joining the points $z_*=x_*+iy_*\in\mathbb{H}$ and gi, $g=\left(\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right)\in SL_2(\mathbb{R})$, is the half-circle of center α and radius r, where

$$|\alpha-z_*|=|\alpha-gi|=r.$$

This gives

$$|\alpha-x_*-iy_*|^2=\left|\alpha-\frac{iA+B}{iC+D}\right|^2=\frac{|i(C\alpha-A)+D\alpha-B|^2}{|iC+D|^2},$$

and after cancelling out the terms containing α^2 we obtain

$$2\alpha(x_*E-F)=(x_*^2+y_*^2)E-G,$$

with

$$E=C^2+D^2,\qquad F=AC+BD,\qquad G=A^2+B^2,$$

leading to

$$\tan\theta_{i,z_*}(g)=\frac{y_*}{x_*-\alpha}=\frac{2y_*(F-x_*E)}{(y_*^2-x_*^2)E+2x_*F-G}.$$

We will keep z_0 and z_1 fixed throughout. Taking

$$z_*=g_0^{-1}z_1=\frac{x_1-x_0+iy_1}{y_0}$$

we have $g_0(x_*+it)=x_1+iy_0t$, $t>0$, so that

$$\theta_{z_0,z_1}(\gamma)=\measuredangle[x_1+i\infty,z_1,\gamma z_0]=\measuredangle[g_0(x_*+i\infty),g_0z_*,g_0gi]=\measuredangle[x_*+i\infty,z_*,gi]=\theta_{i,z_*}(g),$$

and therefore

$$\tan\theta_{z_0,z_1}(\gamma)=\frac{y_*}{x_*-\alpha}=\frac{2y_*(F-x_*E)}{(y_*^2-x_*^2)E+2x_*F-G}. \tag{2.3}$$

When $|A| \le |D|$ we use

$$\left| \frac{F}{E} - \frac{A}{C} \right| = \frac{|D|}{|C|(C^2+D^2)} \le \frac{1}{2C^2},$$
$$\left| \frac{G}{E} - \frac{A^2}{C^2} \right| = \frac{|BC+AD|}{C^2(C^2+D^2)} = \frac{|2AD-1|}{C^2(C^2+D^2)} \le \frac{2}{C^2} + \frac{1}{C^4},$$

to derive

$$\tan\theta_{z_0,z_1}(\gamma) = \frac{2y_*\big(\frac{F}{E} - x_*\big)}{y_*^2 - x_*^2 + 2x_* \frac{F}{E} - \frac{G}{E}} = \frac{2y_*\big(\frac{A}{C} - x_*\big) + O_{z_*}\big(\frac{1}{C^2}\big)}{y_*^2 - \big(\frac{A}{C} - x_*\big)^2 + O_{z_*}\big(\frac{1}{C^2} + \frac{1}{C^4}\big)}. \tag{2.4}$$

When $|D| \le |A|$ we use

$$\left| \frac{F}{G} - \frac{C}{A} \right| = \frac{|B|}{|A|(A^2+B^2)} \le \frac{1}{2A^2},$$
$$\left| \frac{E}{G} - \frac{C^2}{A^2} \right| = \frac{|2AD-1|}{A^2(A^2+B^2)} \le \frac{2}{A^2} + \frac{1}{A^4},$$

to derive

$$\begin{aligned} \tan\theta_{z_0,z_1}(\gamma) &= \frac{2y_*\big(\frac{F}{G} - x_* \frac{E}{G}\big)}{(y_*^2 - x_*^2)\frac{E}{G} + 2x_* \frac{F}{G} - 1} = \frac{2y_*\big(\frac{C}{A} - x_* \frac{C^2}{A^2}\big) + O_{z_*}\big(\frac{1}{A^2} + \frac{1}{A^4}\big)}{(y_*^2 - x_*^2)\frac{C^2}{A^2} + 2x_* \frac{C}{A} - 1 + O_{z_*}\big(\frac{1}{A^2} + \frac{1}{A^4}\big)} \\ &= \frac{2y_*\big(\frac{A}{C} - x_*\big) + O_{z_*}\big(\frac{1}{C^2} + \frac{1}{A^2C^2}\big)}{y_*^2 - \big(\frac{A}{C} - x_*\big)^2 + O_{z_*}\big(\frac{1}{C^2} + \frac{1}{A^2C^2}\big)}. \end{aligned} \tag{2.5}$$

For $\lambda > 0$ set

$$-\alpha_1 := \frac{-1 - \sqrt{1+\lambda^2}}{\lambda} < -1 < 0 < \alpha_2 := \frac{-1 + \sqrt{1+\lambda^2}}{\lambda} = \frac{1}{\alpha_1} < 1.$$

For $\lambda < 0$ set

$$-1 < \alpha_1^* := \frac{1 - \sqrt{1+\lambda^2}}{|\lambda|} < 0 < 1 < \alpha_2^* := \frac{1 + \sqrt{1+\lambda^2}}{|\lambda|} = -\frac{1}{\alpha_1^*}.$$

Letting $\lambda = \tan\omega$, $\omega \in \big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$, we have $\alpha_1 = \cot\frac{\omega}{2}$, $\alpha_2 = \tan\frac{\omega}{2}$ for $\omega > 0$, and $\alpha_1^* = \tan\frac{\omega}{2}$, $\alpha_2^* = -\cot\frac{\omega}{2}$ for $\omega < 0$. A plain calculation gives

$$\frac{2y_*(X - x_*)}{y_*^2 - (X - x_*)^2} < \lambda \iff X - x_* \in \mathfrak{S}(y_*, \lambda), \tag{2.6}$$

with

$$\mathfrak{S}(y_*, \lambda) = \begin{cases} (-\infty, -y_*\alpha_1) \cup (-y_*, y_*\alpha_2) \cup (y_*, \infty) & \text{if } \lambda > 0, \\ (-y_*, 0) \cup (y_*, \infty) & \text{if } \lambda = 0, \\ (-y_*, y_*\alpha_1^*) \cup (y_*, y_*\alpha_2^*) & \text{if } \lambda < 0. \end{cases} \tag{2.7}$$

For fixed $\lambda \in \mathbb{R}$, $z_* \in \mathbb{H}$, and $|\varepsilon_1|$, $|\varepsilon_2|$ small, the roots $X_\pm(\varepsilon_2)$ of $y_*^2 - (X - x_*)^2 + \varepsilon_2 = 0$ and $\widetilde{X}_\pm(\varepsilon_1, \varepsilon_2)$ of $2y_*(X - x_*) + \varepsilon_1 - \lambda\big(y_*^2 - (X - x_*)^2 + \varepsilon_2\big) = 0$ satisfy

$$|X_\pm(\varepsilon_2) - X_\pm(0)| = \left| \sqrt{y_*^2 + \varepsilon_2} - y_* \right| \le \frac{|\varepsilon_2|}{y_*},$$

and respectively

$$\left| \widetilde{X}_\pm(\varepsilon_1, \varepsilon_2) - \widetilde{X}_\pm(0,0) \right| = \frac{|\varepsilon_1 - \lambda\varepsilon_2|}{y_*\sqrt{1+\lambda^2} + \sqrt{y_*^2(1+\lambda^2) - \lambda\varepsilon_1 + \lambda^2\varepsilon_2}} \le \frac{|\varepsilon_1 - \lambda\varepsilon_2|}{y_*\sqrt{1+\lambda^2}} \le \frac{\sqrt{\varepsilon_1^2 + \varepsilon_2^2}}{y_*}.$$

In conjunction with (2.4)–(2.7) this shows, in both cases $|A| \le |D|$ and $|D| \le |A|$, that there is a constant $K_1 = K_1(z_*) > 0$ such that, for any $\gamma \in \Gamma(N)$,

$$\tan\theta_{z_0,z_1}(\gamma) \le \lambda \implies \frac{A}{C} \in x_* + \mathfrak{S}(y_*,\lambda) + \big[-K_1 H(\gamma), K_1 H(\gamma)\big], \tag{2.8}$$

where $H(\gamma) = \frac{1}{C^2} + \frac{1}{A^2C^2} + \frac{1}{C^4}$.

We wish to discard those γ for which one of $|A|$, $|B|$, $|C|$, $|D|$ is small. Note first that, as a result of (2.1), there is a constant $K_0 = K_0(z_0)$ such that $a^2+b^2+c^2+d^2 \le K_0 Q^2$ whenever $A^2+B^2+C^2+D^2 \le Q^2$. For every $K > 0$ let $\mathcal{E}_A(K) = \mathcal{E}_{A,Q,z_0}(K)$ denote the number of $\gamma \in \Gamma(1)$ for which $A^2+B^2+C^2+D^2 \le Q^2$ and $|A| = |a - cx_0| \le K$. Define similarly $\mathcal{E}_B(K)$, $\mathcal{E}_C(K)$, $\mathcal{E}_D(K)$.

Lemma 1. (i) *For every $z_0 \in \mathbb{H}$ and $K \ge 1$*

$$\max\{\mathcal{E}_A(K), \mathcal{E}_C(K), \mathcal{E}_D(K)\} \ll_{z_0} KQ\log Q \qquad (Q \to \infty).$$

(ii). *For every $z_0 \in \mathbb{H}$ and $\alpha \in (0,1)$*

$$\mathcal{E}_B(Q^\alpha) \ll_{z_0} Q^{(3+\alpha)/2}\log Q \qquad (Q\to\infty).$$

Proof. (i) The congruence $bc = 1 \pmod{|a|}$ shows, for fixed c and $a \ne 0$, that the integer b is uniquely determined $\pmod{|a|}$, so it takes $\ll \frac{Q}{|a|}$ values. This gives

$$\mathcal{E}_C(K) \ll 2 + \left(\frac{2K}{y_0}+1\right) \sum_{1\le |a| \le K_0 Q} \frac{Q}{|a|} \ll_{z_0} KQ\log Q.$$

To prove $\mathcal{E}_A(K) \ll_{z_0} KQ\log Q$ note that, for fixed $c \in [-K_0Q, K_0Q]$, there are at most $2K+1$ integers a such that $|a - cx_0| \le K$. For each such a, the congruence $ad = 1 \pmod{|c|}$ uniquely determines $d \pmod{|c|}$, so the number of admissible triples (a,d,b) is $\ll \frac{KQ}{|c|}$, and summing over c we find as above $\mathcal{E}_A(K) \ll_{z_0} KQ\log Q$. The proof of $\mathcal{E}_D(K) \ll_{z_0} KQ\log Q$ is similar.

(ii) Let $\mathcal{E}_A(K)^c$, respectively $\mathcal{E}_D(K)^c$, denote the complement of $\mathcal{E}_A(K)$, respectively $\mathcal{E}_D(K)$, in $\{\gamma \in \Gamma(1) : A^2+B^2+C^2+D^2 \le Q^2\}$. Write $\alpha = 2\alpha' - 1$, $\frac12 < \alpha' < 1$, so that $1+\alpha' = \frac{3+\alpha}{2}$. For every $\gamma \in \mathcal{E}_A(Q^{\alpha'}+1)^c \cap \mathcal{E}_D(Q^{\alpha'}+1)^c$ we have

$$|B| = \frac{|AD-1|}{|C|} > \frac{Q^{2\alpha'}}{Q} = Q^\alpha,$$

showing that $\mathcal{E}_B(Q^\alpha) \subseteq \mathcal{E}_A(Q^{\alpha'}+1) \cup \mathcal{E}_D(Q^{\alpha'}+1)$, and so $\mathcal{E}_D(Q^\alpha) \ll_{z_0} Q^{1+\alpha'}\log Q$. ☐

Note also that

$$\begin{aligned} \left|(A^2+B^2+C^2+D^2) - (C^2+A^2)\left(1+\frac{D^2}{C^2}\right)\right| &= \frac{|AD+BC|}{C^2} = \frac{|2BC+1|}{C^2} \\ &\le \frac{2|B|}{|C|} + \frac{1}{C^2} \ll_{z_0} \frac{Q}{|c|} + \frac{1}{c^2} \ll Q. \end{aligned} \tag{2.9}$$

The relations (2.8) and (2.9) lead us to estimate the number

$$\mathfrak{N}_Q(N,z_0;\beta) := \#\left\{\gamma \in \Gamma(N) : \frac{A}{C} \le \beta,\ (C^2+A^2)\left(1+\frac{D^2}{C^2}\right) \le Q^2\right\} \qquad (Q\to\infty). \tag{2.10}$$

3. Some counting in $\Gamma(N)$

In this section we prove some counting results which will be further used in the proof of Theorem 1 in the next section. Let c and $N \geq 1$ be integers and consider the sum

$$\Phi_N(c) := \sum_{\substack{n|c \\ (n,N)=1}} \frac{\mu(n)}{n}.$$

We first estimate the number

$$\mathcal{N}_{c,N}(I_1 \times I_2) := \#\big\{(a,d) \in I_1 \times I_2 : a \equiv 1,\ d \equiv 1 \pmod N,\ ad \equiv 1 \pmod{Nc}\big\},$$

with fixed N and c, and with a and d in prescribed (short) intervals. The next result extends Lemma 1.6 in [2] from $\Gamma(1)$ to $\Gamma(N)$.

Proposition 1. *For a fixed positive integer N and intervals I_1, I_2 of length less than $|c|$*

$$\mathcal{N}_{c,N}(I_1 \times I_2) = \frac{\Phi_N(c)}{|c|N^2}\,|I_1|\,|I_2| + O_{\varepsilon,N}(|c|^{1/2+\varepsilon}) \qquad (|c| \to \infty).$$

Proof. Replacing (b,c) by $(-b,-c)$ we can assume $c>0$. In this case we write

$$\mathcal{N}_{c,N}(I_1 \times I_2) = \frac{1}{Nc} \sum_{\substack{x \in I_1 \\ (x,Nc)=1 \\ x\equiv 1 \pmod N}} \sum_{\substack{y \in I_2 \\ y \equiv 1 \pmod N}} \sum_{k \pmod{Nc}} e\left(\frac{k(y-\bar{x})}{Nc}\right) = \mathcal{M} + \mathcal{E},$$

where $\bar{x}$ is the multiplicative inverse of $x \pmod{Nc}$ and $e(t) = \exp(2\pi i t)$. The contribution

$$\mathcal{M} = \frac{1}{Nc} \sum_{\substack{x \in I_1 \\ (x,Nc)=1 \\ x\equiv 1 \pmod N}} \sum_{\substack{y \in I_2 \\ y \equiv 1 \pmod N}} \sum_{0 \leq \ell < N} e\left(\frac{\ell(y-\bar{x})}{N}\right) \tag{3.1}$$

of terms with $c \mid k$ to $\mathcal{N}_{c,N}(I_1, I_2)$ will be treated as a main term, while the contribution

$$\mathcal{E} = \frac{1}{Nc} \sum_{\substack{0 \leq k < Nc \\ c \nmid k}} \sum_{\substack{y \in I_2 \\ y \equiv 1 \pmod N}} e\left(\frac{ky}{Nc}\right) \sum_{\substack{x \in I_1 \\ (x,Nc)=1 \\ x\equiv 1 \pmod N}} e\left(-\frac{k\bar{x}}{Nc}\right) \tag{3.2}$$

of terms with $c \nmid k$ will be treated as an error term.

To estimate $\mathcal{E}$ consider for I interval and $q \in \mathbb{N}$, $m,n \in \mathbb{Z}$, the incomplete Kloosterman sum

$$S_I(m,n;q) := \sum_{\substack{a \in I \\ (a,q)=1}} e\left(\frac{ma+n\bar{a}}{q}\right),$$

where $\bar{a}$ is the multiplicative inverse of $a \pmod q$. The complete Kloosterman sum $S(m,n;q)$ is just $S_{[0,q-1]}(m,n;q)$. For any interval $I \subseteq [0,q-1]$ and integers m,n, not both divisible by q, the Weil bound on Kloosterman sums leads (cf., e.g., [2, Lemma 1.6]) to

$$|S_I(m,n;q)| \ll_\varepsilon (n,q)^{1/2} q^{1/2+\varepsilon}. \tag{3.3}$$

Writing now the inner sum in (3.2) as

$$\sum_{\substack{x \in I_1 \\ (x,Nc)=1}} e\left(-\frac{k\bar{x}}{Nc}\right) \frac{1}{N} \sum_{s \pmod N} e\left(\frac{s(x-1)}{N}\right) = \frac{1}{N} \sum_{s \pmod N} e\left(-\frac{s}{N}\right) S_{I_1}(cs,-k;Nc)$$

and applying (3.3) we find

$$|\mathcal{E}| \ll_\varepsilon \frac{(Nc)^{1/2+\varepsilon}}{Nc} \sum_{\substack{0\le k<Nc \\ c\nmid k}} (k,Nc)^{1/2} \Bigg| \sum_{\substack{y\in I_2 \\ y\equiv 1 \pmod{N}}} e\Big(\frac{ky}{Nc}\Big)\Bigg|.$$

Treating the inner sum above as a geometric progression of ratio $e(\frac{k}{c})$ and using the inequality $|\sin \pi t| \ge 2\|t\| = 2\,\mathrm{dist}(t,\mathbb{Z})$, $t\in\mathbb{R}$, the inner sum above is $\le \min\big\{|I_2|, \frac{1}{2\|k/c\|}\big\}$. Employing also the inequality $(k,Nc)\le (k,c)N$ we further find

$$\begin{aligned} |\mathcal{E}| &\ll_\varepsilon N^{1+\varepsilon} \frac{c^{1/2+\varepsilon}}{c} \sum_{0<\ell<c} \frac{(\ell,c)^{1/2}}{2\|\frac{\ell}{c}\|} \ll N^{1+\varepsilon} c^{-1/2+\varepsilon} \sum_{d\mid c} \sum_{m\le \frac{c}{2d}} d^{1/2}\,\frac{c}{dm} \\ &\le N^{1+\varepsilon} c^{1/2+\varepsilon} \sum_{d\mid c} d^{-1/2}\log c \ll_{\varepsilon,N} c^{1/2+2\varepsilon}. \end{aligned}$$

Concerning the main term $\mathcal{M}$, from $x\bar{x}=1 \,(\mathrm{mod}\ N)$ and $x=1\,(\mathrm{mod}\ N)$ we infer $\bar{x}=1$ $(\mathrm{mod}\ N)$, and so $N \mid (y-\bar{x})$. The inner sum in (3.1) is equal to N and we get

$$\mathcal{M} = \frac{1}{c} \sum_{\substack{x\in I_1 \\ (x,Nc)=1 \\ x\equiv 1 \pmod{N}}} 1 \sum_{\substack{y\in I_2 \\ y\equiv 1 \pmod{N}}} 1 = \frac{1}{c}\left(\frac{|I_2|}{N}+O(1)\right) \sum_{\substack{x\in I_1 \\ (x,Nc)=1 \\ x\equiv 1 \pmod{N}}} 1.$$

Using $x=1\,(\mathrm{mod}\ N)$ and Möbius summation, the latter sum above can also be expressed as

$$\begin{aligned} \sum_{\substack{x\in I_1 \\ x\equiv 1 \pmod{N}}} \sum_{\substack{d\mid x \\ d\mid c}} \mu(d) &= \sum_{\substack{x\in I_1 \\ x\equiv 1 \pmod{N}}} \sum_{\substack{d\mid x,\, d\mid c \\ (d,N)=1}} \mu(d) = \sum_{\substack{d\mid c \\ (d,N)=1}} \mu(d) \sum_{\substack{x\in I_1,\, d\mid x \\ x\equiv 1 \pmod{N}}} 1 \\ &= \sum_{\substack{d\mid c \\ (d,N)=1}} \mu(d)\left(\frac{|I_1|}{dN}+O(1)\right) = \frac{|I_1|}{N}\,\Phi_N(c) + O_\varepsilon(c^\varepsilon), \end{aligned}$$

which completes the proof. □

Denote by $V_I(f)$ the total variation of a function f defined on the interval I.

Corollary 3. *For I interval, C^1 functions $f_1,f_2 : I\to\mathbb{R}$ with $f_1\le f_2$, and $T\ge 1$ integer, the cardinality $\mathcal{N}_{c,N}(f_1,f_2)$ of the set*

$$\big\{(a,d)\in\mathbb{Z}^2 : d\in I,\ f_1(d)\le a\le f_2(d),\ a\equiv 1,\ d\equiv 1 \pmod{N},\ ad\equiv 1 \pmod{Nc}\big\}$$

can be expressed as

$$\mathcal{N}_{c,N}(f_1,f_2) = \frac{\Phi_N(c)}{|c|N^2}\int_I (f_2-f_1) + \mathcal{E}_{c,N,f_1,f_2} \qquad (|c|\to\infty),$$

with

$$\mathcal{E}_{c,N,f_1,f_2} \ll_{\varepsilon,N} \frac{|I|}{T|c|}\big(V_I(f_1)+V_I(f_2)\big) + T|c|^{1/2+\varepsilon}\left(1+\frac{|I|}{T|c|}\right)\left(1+\frac{\|f_1\|_\infty+\|f_2\|_\infty}{|c|}\right).$$

Proof. This follows from Proposition 1 as in the proof of [1, Lemma 3.1]. □

Lemma 2. *For every interval J and every C^1 function $f : J\to\mathbb{R}$*

$$\sum_{c\in J} \Phi_N(c) f(c) = C_N \int_J f + O\Big(\big(\|f\|_\infty + V_J(f)\big)\log \sup_{\xi\in J}|\xi|\Big),$$

with C_N as in (1.3).

Proof. We can assume without loss of generality that $J=(0,Q]$. For each $n\geq 1$ consider the n-dilate function $f_n(x):=f(nx)$, $x\in[0,\frac{Q}{n}]$, for which $\|f_n\|_\infty=\|f\|_\infty$, $\int_0^{Q/n} f_n=\int_0^Q f$, and $V_0^{Q/n}(f_n)=V_0^Q(f)$. Using Möbius and Euler-MacLaurin summation we get

$$\begin{aligned}\sum_{c=1}^{Q}\Phi_N(c)f(c) &= \sum_{c=1}^{Q}\sum_{\substack{n\mid c\\ (n,N)=1}}\frac{\mu(n)}{n}f(c)=\sum_{\substack{n\leq Q\\ (n,N)=1}}\frac{\mu(n)}{n}\sum_{c\leq Q/n} f_n(c)\\ &= \sum_{\substack{n\leq Q\\ (n,N)=1}}\frac{\mu(n)}{n}\left(\int_0^{Q/n} f_n+O\big(\|f_n\|_\infty+V_0^{Q/n}(f)\big)\right)\\ &= \left(\sum_{\substack{n\geq 1\\ (n,N)=1}}\frac{\mu(n)}{n^2}+O\Big(\frac{1}{Q}\Big)\right)\int_0^Q f+O\Big(\log Q\big(\|f\|_\infty+V_0^Q(f)\big)\Big)\\ &= C_N\int_0^Q f+O\Big(\log Q\big(\|f\|_\infty+V_0^Q(f)\big)\Big),\end{aligned}$$

which represents the desired conclusion. $\square$

Corollary 4. *For every interval I and every C^1 function $f:I\to\mathbb{R}$*

$$\sum_{\substack{c\in I\\ N\mid c}}\Phi_N(c)f(c)=\frac{C_N}{N}\int_I f+O\Big(\big(\|f\|_\infty+V_I(f)\big)\log\sup_{\xi\in I}|\xi|\Big).$$

Proof. Apply Lemma 2 to $J=\frac{1}{N}I$, $f_N(x)=f(Nx)$, using $\Phi_N(Nc')=\Phi_N(c')$, $\int_J f_N=\frac{1}{N}\int_I f$, $\|f_N\|_\infty=\|f\|_\infty$, and $V_I(f_N)=V_J(f)$. $\square$

4. Proof of the main results

We first estimate the quantity defined in (2.10).

Proposition 2. *For every positive integer N and every $z_0\in\mathbb{H}$, $\beta\in[-\infty,\infty]$, as $Q\to\infty$,*

$$\mathfrak{N}_Q(N,z_0;\beta)=\frac{\pi(\pi+2\arctan\beta)C_N}{2N^3}Q^2+O_{\varepsilon,N,z_0}(Q^{7/4+\varepsilon}).$$

Proof. Define

$$\begin{aligned} I_c &= cx_0+\begin{cases}\big[-\sqrt{Q^2-c^2y_0^2},\ \min\{\beta cy_0,\sqrt{Q^2-c^2y_0^2}\ \}\big] & \text{if } c\in[0,Q/y_0]\,,\\ \big[\max\{\beta cy_0,-\sqrt{Q^2-c^2y_0^2}\ \},\sqrt{Q^2-c^2y_0^2}\ \big] & \text{if } c\in[-Q/y_0,0]\,,\end{cases}\\ f(c,a) &= |c|y_0\sqrt{\frac{Q^2}{c^2y_0^2+(a-cx_0)^2}-1},\\ f_1(c,a) &= -cx_0-f(c,a),\quad f_2(c,a)=-cx_0+f(c,a),\qquad c\in[-Q/y_0,Q/y_0]\,,\ a\in I_c,\\ F(c) &= F_{z_0,\beta}(c)=\frac{2}{|c|}\int_{I_c} f(c,a)\ da.\end{aligned}$$

Writing the inequalities from (2.10) as

$$\begin{cases}|C|\leq Q,\\ -\sqrt{Q^2-C^2}\leq A\leq\sqrt{Q^2-C^2}\\ -|C|\sqrt{\frac{Q^2}{C^2+A^2}-1}\leq D\leq |C|\sqrt{\frac{Q^2}{C^2+A^2}-1},\end{cases}\quad\text{and}\quad \begin{cases}A\leq\beta C & \text{if } C>0,\\ A\geq\beta C & \text{if } C<0,\end{cases}$$

and using (2.1) we gather

$$\begin{aligned}\mathfrak{N}_Q(N,z_0;\beta) &= \#\Big\{\gamma\in\Gamma(N) : |c|y_0\le Q,\ a\in I_c,\ d\in[f_1(c,a),f_2(c,a)]\Big\}\\ &= \sum_{|c|\le Q/y_0}\mathcal{N}_{c,N}\big(f_1(c,\cdot),f_2(c,\cdot)\big).\end{aligned}\tag{4.1}$$

Note that $\max\big\{\|f(c,\cdot)\|_\infty, V_{I_c}\big(f(c,\cdot)\big)\big\}\ll Q$ on I_c, thus Corollary 3 with $T=[Q^{1/4}]$ gives

$$\mathcal{N}_{c,N}\big(f_1(c,\cdot),f_2(c,\cdot)\big)=\frac{1}{N^2}\Phi_N(c)F(c)+\mathcal{E}_{c,N},\tag{4.2}$$

with

$$\mathcal{E}_{c,N}\ll_{\varepsilon,N} Q^{7/4}|c|^{-1}+Q^{5/4}|c|^{-1/2+\varepsilon}+Q^2|c|^{-3/2+\varepsilon}.\tag{4.3}$$

Fix some constant $\alpha\in\big[\frac12,\frac34\big]$. The relation $bc\equiv -1 \pmod{|a|}$ and the constraint $|a|\ll_{z_0} Q$ give the trivial estimate

$$\sum_{|c|\le Q^\alpha}\mathcal{N}_{c,N}\big(f_1(c,\cdot),f_2(c,\cdot)\big)\ll_{z_0}\sum_{1\le|a|\le Q}Q^\alpha\frac{Q}{|a|}\ll Q^{1+\alpha}\log Q\ll_\varepsilon Q^{7/4+\varepsilon}.\tag{4.4}$$

On the other hand (4.3) leads to

$$\begin{aligned}\sum_{Q^\alpha<|c|\le Q/y_0}\mathcal{E}_{c,N} &\ll_{\varepsilon,z_0,N} Q^{7/4}\log Q+Q^{5/4}\sum_{1\le c\le Q}c^{-1/2+\varepsilon}+Q^2\sum_{c>Q^\alpha}c^{-3/2+\varepsilon}\\ &\ll_\varepsilon Q^{7/4+\varepsilon}+Q^{5/4+1/2+\varepsilon}+Q^{2+\alpha(-1/2+\varepsilon)}\ll Q^{7/4+\varepsilon}.\end{aligned}\tag{4.5}$$

From (4.1)–(4.5) we now infer

$$\mathfrak{N}_Q(N,z_0;\beta)=\frac{1}{N^2}\sum_{Q^\alpha\le|c|\le Q/y_0}\Phi_N(c)F(c)+O_{\varepsilon,N,z_0}(Q^{7/4+\varepsilon}).\tag{4.6}$$

Using $I_c\subseteq\big[-\sqrt{Q^2-c^2y_0^2},\sqrt{Q^2-c^2y_0^2}\,\big]$ and the change of variable $u=C\tan x$ we get

$$\begin{aligned}F(c) &= 2y_0\int_{I_c}\sqrt{\frac{Q^2}{c^2y_0^2+(a-cx_0)^2}-1}\,da\le 4y_0\int_0^{\sqrt{Q^2-C^2}}\sqrt{\frac{Q^2}{C^2+u^2}-1}\,du\\ &= 4y_0\int_0^{\arctan\sqrt{Q^2/C^2-1}}\sqrt{Q^2-\frac{C^2}{\cos^2 x}}\,\frac{dx}{\cos x}\le 4y_0Q\int_0^{\arctan\sqrt{Q^2/C^2-1}}\frac{dx}{\cos x}\\ &= 2y_0Q\log\frac{1+\sin x}{1-\sin x}\,\Bigg|_{x=0}^{\arctan\sqrt{Q^2/C^2-1}}=4y_0Q\log\left(\frac{Q}{C}+\sqrt{\frac{Q^2}{C^2}-1}\,\right)\ll_{z_0} Q\log Q.\end{aligned}$$

The total variation of F on $\big[-\frac{Q}{y_0},-Q^\alpha\big]$ and on $\big[Q^\alpha,\frac{Q}{y_0}\big]$ is also $\ll_{z_0} Q\log Q$ because F is slowly oscillating. Applying Corollary 4 to the sum from (4.6) we now infer

$$\begin{aligned}\mathfrak{N}_Q(N,z_0;\beta) &= \frac{C_N}{N^3}\int_{Q^\alpha\le|c|\le Q/y_0}F(c)\,dc+O_{\varepsilon,N,z_0}(Q^{7/4+\varepsilon})\\ &= \frac{C_N}{N^3}\int_{-Q/y_0}^{Q/y_0}F(c)\,dc+O_{\varepsilon,N,z_0}(Q^{7/4+\varepsilon}).\end{aligned}$$

Using the substitution $c=\frac{Qu}{y_0}$, $a=Qv+cx_0=\big(v+\frac{ux_0}{y_0}\big)Q$, the integral in the main term above is evaluated as

$$\begin{aligned}
\int_{-Q/y_0}^{Q/y_0} F(c)\,dc &= 2\int_{-Q/y_0}^{Q/y_0}\int_{I_c} f(c,a)\,da\,dc \\
&= 2\iint_{\substack{u^2+v^2\leq 1\\ u\geq 0,\, v\leq \beta u}} \sqrt{\frac{1}{u^2+v^2}-1}\;du\,dv + 2\iint_{\substack{u^2+v^2\leq 1\\ u\leq 0,\, v\geq \beta u}} \sqrt{\frac{1}{u^2+v^2}-1}\;du\,dv \\
&= 2\int_0^1\int_{-\pi/2}^{\arctan\beta}\sqrt{1-r^2}\;d\theta\,dr + 2\int_0^1\int_{\pi/2}^{\pi+\arctan\beta}\sqrt{1-r^2}\;d\theta\,dr \\
&= \frac{\pi(\pi+2\arctan\beta)}{2}.
\end{aligned}$$

This completes the proof of the proposition. □

Taking stock on (2.9) we obtain (recall that $Q^2=e^R+O(e^{-R})$)

$$\begin{aligned}
\#\Gamma(N)_{z_0,R} &= \#\left\{\gamma\in\Gamma(N): A^2+B^2+C^2+D^2\leq Q^2\right\} = \mathfrak{N}_{\sqrt{Q^2+O_{z_0}(Q)}}(N,z_0;\infty) \\
&= \frac{\pi^2 C_N Q^2}{N^3} + O_{\varepsilon,N,z_0}(Q^{7/4+\varepsilon}) = \frac{6Q^2}{[\Gamma(1):\Gamma(N)]} + O_{\varepsilon,N,z_0}(Q^{7/4+\varepsilon}) \\
&= \frac{6e^R}{[\Gamma(1):\Gamma(N)]} + O_{\varepsilon,N,z_0}\left(e^{(7/8+\varepsilon)R}\right),
\end{aligned} \tag{4.7}$$

which proves Corollary 2.

Proof of Theorem 1. Set $\mathfrak{N}_Q(\beta)=\mathfrak{N}_Q(N,z_0;\beta)$. As a consequence of Proposition 2 and of the inequality $|\arctan(\beta+\beta_0)-\arctan\beta|\leq|\beta_0|$ we have

$$|\mathfrak{N}_Q(\beta+\beta_0)-\mathfrak{N}_Q(\beta)| \ll_{\varepsilon,N,z_0} Q^2|\beta_0| + Q^{7/4+\varepsilon}. \tag{4.8}$$

Let $S_Q(\omega)=S_Q(N,z_0,z_1;\omega)$ denote the cardinality of the set of $\gamma\in\Gamma(N)$ with $A^2+B^2+C^2+D^2\leq Q^2$ and $-\frac{\pi}{2}\leq\theta_{z_0,z_1}(\gamma)\leq\omega$. Partitioning this set according to whether or not $\min\{|A|,|C|\}>Q^\alpha$ and employing Lemma 1 we find that, up to an error $\ll_{z_0} Q^{1+\alpha}\log Q$, $S_Q(\omega)$ equals

$$\#\left\{\gamma\in\Gamma(N): A^2+B^2+C^2+D^2\leq Q^2, |A|,|C|>Q^\alpha, -\pi/2\leq\theta_{z_0,z_1}(\gamma)\leq\omega\right\}. \tag{4.9}$$

By (2.9) there is $K_2=K_2(z_0)>0$ such that the number in (4.9) is

$$\leq \#\left\{\gamma\in\Gamma(N): (C^2+A^2)\left(1+\frac{D^2}{C^2}\right)\leq Q_1^2, |A|,|C|>Q^\alpha,\ -\frac{\pi}{2}\leq\theta_{z_0,z_1}(\gamma)\leq\omega\right\}, \tag{4.10}$$

where we set $Q_1:=\sqrt{Q^2+K_2Q}=Q+O_{z_0}(1)$. According to (2.8) the number in (4.10) is

$$\leq \#\left\{\gamma\in\Gamma(N): (C^2+A^2)\left(1+\frac{D^2}{C^2}\right)\leq Q_1^2,\ \frac{A}{C}\in x_*+\mathfrak{S}(y_*,\tan\omega)+\left[-\frac{3K_1}{Q^{2\alpha}},\frac{3K_1}{Q^{2\alpha}}\right]\right\}.$$

Taking $\alpha=\frac{1}{8}$ and applying (4.8) to $|\beta_0|=Q^{-2\alpha}=Q^{-1/4}$ we find

$$\begin{aligned}
S_Q(\omega) \leq &\#\left\{\gamma\in\Gamma(N): (C^2+A^2)\left(1+\frac{D^2}{C^2}\right)\leq Q_1^2,\ \frac{A}{C}\in x_*+\mathfrak{S}(y_*,\tan\omega)\right\} \\
&+ O_{\varepsilon,N,z_0,z_1}(Q^{7/4+\varepsilon}).
\end{aligned} \tag{4.11}$$

The number of matrices $\gamma \in \Gamma(N)$ for which $\frac{A}{C} = \mu$ and $A^2+B^2+C^2+D^2 \leq Q^2$ is $\ll_{z_0,\mu} Q$ as $Q \to \infty$. Using this fact together with (2.7), (2.9), and (2.10), we find that, up to a term of order $O_{z_0}(Q_1) = O_{z_0}(Q)$, the main term in the right-hand side of (4.11) is given by

$$\begin{aligned}
&\begin{cases}
\mathfrak{N}_{Q_1}\left(x_*-y_*\cot\frac{\omega}{2}\right)+\mathfrak{N}_{Q_1}\left(x_*+y_*\tan\frac{\omega}{2}\right)-\mathfrak{N}_{Q_1}(x_*-y_*) & \\
\qquad +\mathfrak{N}_{Q_1}(\infty)-\mathfrak{N}_{Q_1}(x_*+y_*) & \text{if } \omega>0,\\
\mathfrak{N}_{Q_1}(x_*)-\mathfrak{N}_{Q_1}(x_*-y_*)+\mathfrak{N}_{Q_1}(\infty)-\mathfrak{N}_{Q_1}(x_*+y_*) & \text{if } \omega=0,\\
\mathfrak{N}_{Q_1}\left(x_*+y_*\tan\frac{\omega}{2}\right)-\mathfrak{N}_{Q_1}(x_*-y_*) & \\
\qquad +\mathfrak{N}_{Q_1}\left(x_*-y_*\cot\frac{\omega}{2}\right)-\mathfrak{N}_{Q_1}(x_*+y_*) & \text{if } \omega<0,
\end{cases}\\
&\quad=\mathfrak{N}_{Q_1}\left(x_*+y_*\tan\frac{\omega}{2}\right)+\mathfrak{N}_{Q_1}\left(x_*-y_*\cot\frac{\omega}{2}\right)-\mathfrak{N}_{Q_1}(x_*+y_*)-\mathfrak{N}_{Q_1}(x_*-y_*)\\
&\qquad\quad+\begin{cases}\mathfrak{N}_{Q_1}(\infty) & \text{if } \omega>0,\\ 0 & \text{if } \omega<0.\end{cases}
\end{aligned}\tag{4.12}$$

As a result of Proposition 2 and $Q_1 = Q + O_{z_0}(1)$ the expression in (4.12) equals

$$\begin{aligned}
\frac{\pi C_N Q^2}{N^3}\Bigg(&\arctan\left(x_*+y_*\tan\frac{\omega}{2}\right)+\arctan\left(x_*-y_*\cot\frac{\omega}{2}\right)-\arctan(x_*+y_*)\\
&-\arctan(x_*-y_*)+\begin{cases}\pi & \text{if } \omega>0,\\ 0 & \text{if } \omega<0,\end{cases}\Bigg)+O_{\varepsilon,N,z_0,z_1}(Q^{7/4+\varepsilon}).
\end{aligned}$$

Letting Ξ_{x_*,y_*} as in (1.4) we now infer

$$S_Q(\omega) \leq \frac{\pi^2 C_N \Xi_{x_*,y_*}(\omega)}{N^3} Q^2 + O_{\varepsilon,N,z_0,z_1}(Q^{7/4+\varepsilon}). \tag{4.13}$$

The opposite inequality

$$S_Q(\omega) \geq \frac{\pi^2 C_N \Xi_{x_*,y_*}(\omega)}{N^3} Q^2 + O_{\varepsilon,N,z_0,z_1}(Q^{7/4+\varepsilon})$$

is derived in a similar way. Therefore equality holds in (4.13). Equality (1.5) now follows taking $Q^2 = 2\cosh R = e^R + e^{-R}$.

Estimates (1.5) and (4.7) provide

$$\begin{aligned}
\mathbb{P}_{\Gamma(N),B(z_0,R),z_0,z_1}(\omega) &= \frac{\#\{\gamma\in\Gamma(N)_{z_0,R} : -\pi/2 \leq \theta_{z_0,z_1}(\gamma) \leq \omega\}}{\#\Gamma(N)_{z_0,R}}\\
&= \frac{\frac{\pi^2 C_N}{N^3}\Xi_{x_*,y_*}(\omega)e^R + O_{\varepsilon,N,z_0,z_1}(e^{(7/8+\varepsilon)R})}{\frac{\pi^2 C_N}{N^3}e^R + O_{\varepsilon,N,z_0,z_1}(e^{(7/8+\varepsilon)R})}\\
&= \Xi_{x_*,y_*}(\omega) + O_{\varepsilon,N,z_0,z_1}\left(e^{(-1/8+\varepsilon)R}\right).
\end{aligned}\tag{4.14}$$

The function Ξ_{x_*,y_*} is differentiable on $[-\frac{\pi}{2},\frac{\pi}{2}]$ with

$$\begin{aligned}
\Xi'_{x_*,y_*}(\omega) &= \frac{y_*}{2\pi\cos^2\frac{\omega}{2}\big(1+(x_*+y_*\tan\frac{\omega}{2})^2\big)} + \frac{y_*}{2\pi\sin^2\frac{\omega}{2}\big(1+(x_*-y_*\cot\frac{\omega}{2})^2\big)} \\
&= \frac{y_*}{2\pi}\left(\frac{1}{\cos^2\frac{\omega}{2}+\big(x_*\cos\frac{\omega}{2}+y_*\sin\frac{\omega}{2}\big)^2} + \frac{1}{\sin^2\frac{\omega}{2}+\big(x_*\sin\frac{\omega}{2}-y_*\cos\frac{\omega}{2}\big)^2}\right) \\
&= \frac{2}{\pi}\cdot\frac{y_*(1+x_*^2+y_*^2)}{(1+x_*^2+y_*^2)^2-\big((1+x_*^2-y_*^2)\cos\omega+2x_*y_*\sin\omega\big)^2} \\
&= \frac{1}{\pi}\varrho_{z_0,z_1}(\omega).
\end{aligned} \tag{4.15}$$

The second part of Theorem 1 now follows from (4.14) and (4.15). □

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